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INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

# LATTICE POINTS ON THE HOMOGENEOUS CONE $z^2 = 10x^2 - 6y^2$

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## Abstract

The Cone represented by the homogeneous ternary quadratic equation given by $z^2 = 10x^2 - 6y^2$. is considered for determining non-zero distinct integral points on it. Four different patterns of solutions, each satisfying the cone under consideration are illustrated. A few interesting properties between the integral solutions on the cone and special number patterns namely,polygonal numbers, star numbers, woodall numbers , pentatope numbers are exhibited. Also, knowing an integral point on the given cone, three different triples of the non-zero distinct integers , each satisfying the given homogeneous cone are obtained.

**Keywords :**Ternary Quadratic, Lattice Points, Homogeneous Cone. MSC2000Subject classification: 11D09

**Notations:**

| Special numbers | Notations | Definitions |
|---|---|---|
| Star number | $S_n$ | $6n(n-1)+1$ |
| Regular Polygonal number | $t_{m,n}$ | $n\left[1+\frac{(n-1)(m-2)}{2}\right]$ |
| Pronic number | $P_n$ | $n(n+1)$ |
| Decagonal number | $D_n$ | $n(4n-3)$ |
| Tetra decagonal number | $TD_n$ | $\frac{n(12n-10)}{2}$ |
| Tetrahedral number | $TH_n$ | $\frac{n(n+1)(n+2)}{6}$ |
| Pentatope number | $PT_n$ | $\frac{n(n+1)(n+2)(n+3)}{24}$ |
| Centered Pentagonal number | $CP_n$ | $\frac{1}{2}[5n^2+5n+2]$ |
| HauyRhombicdodecahedral number | $HRD_n$ | $(2n-1)(8n^2-14n+7)$ |
| Triangular number | $T_n$ | $\frac{n(n+1)}{2}$ |
| Woodall number | $W_n$ | $n2^n-1$ |
| Centered Hex number | $Ct_{6,n}$ | $3n^2+3n+1$ |

## Introduction

The ternary quadratic diophantine equations (homogeneous and non-homogeneous)offer an unlimited field for research by reason of variety [1-2]. For an extensive review of various problems one may refer [3-17]. This communication concerns with yet another interesting ternary quadratic equation representing a homogeneous cone $z^2 = 10x^2 - 6y^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations between the solutions and special number patterns are presented. Further three different general forms for generating sequence of integral points based on the given point on the considered cone are exhibited.

## Method of Analysis

The Ternary quadratic equation representing homogeneous equation is

$$z^2 = 10x^2 - 6y^2 \quad (1)$$

To start with, it is seen that (1) is satisfied by the following triples:
(22,26,28), (14,18,4), (29,37,14), (35,43,34), (73,81,118), (31,39,22), (44,52,56), (55,71,2), (28,36,8), (58,74,28)
However, we have other patterns of solutions which are illustrated as follows.

**Method I:**

Introducing the linear transformations,

$$x = X + 6T,\ \ y = X + 10T,\ \ z = 2W \quad (2)$$

in (1), it is written as

$$W^2 = X^2 - 6oT^2 \quad (3)$$

which is satisfied by

$$X = a^2 + 60b^2,\ \ T = 2ab,\ \ W = a^2 - 60b^2 \quad (4)$$

From (4) and (2) the non-zero distinct integral points on the homogeneous cone(1) are given by,

$$x = x(a,b) = a^2 + 60b^2 + 12ab$$
$$y = y(a,b) = a^2 + 60b^2 + 20ab$$
$$z = z(a,b) = 2a^2 - 120b^2$$

**Properties:**

1. $x(a,b)$ *is written as sum of perfect square and a nasty number.*
2. $x(a,b) + y(a,b) + z(a,b)$ *is written as difference between two perfect square.*
3. $3(x(a,b) + y(a,b))$ *is written as difference between two nasty numbers.*
4. $10x(a,b) + a^2 z(a,b) - z(a,b) - 4T_{a^2} + 120P_{ab-1}$ *is a nasty number* *when* $a = 120r^2 - s^2$ *and* $b = 2rs$
5. $3(z(a,1) + W_4 - 55$ *is a nasty number.*
6. $x(a,1) + y(a,1) + z(a,1) \equiv 0 (mod\ 5)$
7. $x(a,1) + 2(Ct_a - CP_a) + t_{8,a} \equiv 0\ (mod\ 2)$
8. $z(a,1) + W_4 + t_{8,a} - t_{6,a} - P_a - 2P_{a-1} \equiv 1\ (mod\ 2)$
9. $y(a,1) + TD_a - t_{28,a} \equiv 5\ (mod\ 11)$
10. $z(a,1) + W_4 + D_a - S_a \equiv 0\ (mod\ 3)$

**Method II:**

Equation (1) can be written as

$$X^2 - 60T^2 = W^2 * 1 \quad (5)$$

$$Let\ W = a^2 - 60b^2,\ \ a,b \neq 0 \quad (6)$$

Write 1 as

$$1 = \frac{(8+\sqrt{60})(8-\sqrt{60})}{4} \quad (7)$$

Using (6) and (7) and employing the method of factorization, define

$$(X + \sqrt{60}T)(X - \sqrt{60}T) = \frac{(8+\sqrt{60})(8-\sqrt{60})}{4}(a+\sqrt{60}b)^2(a-\sqrt{60}b)^2$$

Equating rational and irrational parts, we get

$$X = \frac{1}{2}[8a^2 + 480b^2 + 120ab]$$

$$T = \frac{1}{2}[(a^2 + 60b^2 + 16ab]$$

*Substituting the values of X, T and W in (2), the non − zero distinct integral points satisfying the homogeneous cone is given by*

$$x = x(a,b) = 7a^2 + 420b^2 + 108ab$$
$$y = y(a,b) = 9a^2 + 540b^2 + 140ab$$
$$z = z(a,b) = 2a^2 - 120b^2$$

**Properities:**

1. $y(a,b) + z(a,b) - x(a,b)$ *is written as difference between two perfect squares.*
2. $2\big(y(a,b) - x(a,b)\big)$ *is written as the difference between two perfect numbers.*
3. $3\big(y(a,b) - x(a,b)\big)$ *is written as the difference between two nasty numbers.*
4. $y(a,b) + 4ab$ *is written as difference between two perfect squares.*
5. $y(a,1) + 4a + 36$ *is a perfect square.*
6. $x(a,1)z(a,1) - 336PT_n - HRD_n + W_{12} \equiv 0\ (mod\ 2).$
7. $42TH_3 + 7t_{32,a} + 2t_{5,a} - x(a,1) \equiv 1(mod\ 2).$
8. $x(2,b) - 28\big(t_{32,a} + P_b\big) \equiv 2(mod\ 5).$

**Method III:**

Write 1 as

$$1 = \frac{(16+\sqrt{60})(16-\sqrt{60})}{196} \qquad (8)$$

Using (8) and (6) in (5) and performing an analysis similar to Method II, the corresponding non-zero distinct integral points on the homogeneous cone (1) are given by

$$x = x(a,b) = 14(22a^2 + 1320b^2 + 312ab)$$
$$y = y(a,b) = 14(26a^2 + 1560b^2 + 440ab)$$
$$z = z(a,b) = 196(2a^2 - 120b^2)$$

**Remark:**

It is to be noted that instead of (2) one may consider the linear transformation

$$x = X - 6T,\ \ y = X - 10T,\ \ z = 2W$$

For this choice, the corresponding three sets of integral solutions to (1) by following the above three methods are respectively represented as below,

SET 1:

$$x = x(a,b) = a^2 + 60b^2 - 12ab$$
$$y = y(a,b) = a^2 + 60b^2 - 20ab$$
$$z = z(a,b) = 2a^2 - 120b^2$$

SET 2:

$$x = x(a,b) = a^2 + 60b^2 + 12ab$$
$$y = y(a,b) = -(a^2 + 60b^2 + 20ab)$$
$$z = z(a,b) = 2a^2 - 120b^2$$

SET 3:

$$x = x(a,b) = 14(10a^2 + 600b^2 - 72ab)$$
$$y = y(a,b) = 14(6a^2 + 360b^2 - 200ab)$$
$$z = z(a,b) = 196(2a^2 - 120b^2)$$

**Method 4:**

Equation (1) can be written as

$$W^2 + 60T^2 = X^2 * 1 \qquad (9)$$

$$Let\ X = a^2 + 60b^2, \qquad a, b \neq 0 \qquad (10)$$

Write 1 as,

$$1 = \frac{(2 + i\sqrt{60})(2 - i\sqrt{60})}{64} \qquad (11)$$

Using (10) and (11) and employing the method of factorization, define

$$\left(\frac{2 + i\sqrt{60}}{8}\right)\left(\frac{2 - i\sqrt{60}}{8}\right)(a + i\sqrt{60})^2(a - i\sqrt{60})^2 = (W + i\sqrt{60}T)(W - i\sqrt{60}T)$$

Equating real and imaginary parts, we get

$$W = \frac{1}{8}[2(a^2 - 60b^2) - 120ab]$$

$$T = \frac{1}{8}[a^2 - 60b^2 + 4ab]$$

*Substituting the values of X,T and W in (2), the non − zero distinct integral points satisfying the homogeneous cone is given by*

$$x = x(a, b) = 7a^2 + 15b^2 + 6ab$$

$$y = y(a, b) = 9a^2 - 15b^2 + 10ab$$

$$z = z(a, b) = 2a^2 - 30b^2 - 60ab$$

It is noted that in (2) we may also take,

$$x = X - 6T, \quad y = X - 10T, \quad z = 2W$$

For this choice, the corresponding integral points on (1) are obtained as,

$$x = x(a, b) = a^2 + 105b^2 - 6ab$$

$$y = y(a, b) = -a^2 + 135b^2 - 10ab$$

$$z = z(a, b) = 2a^2 - 30b^2 - 60ab$$

**RemarkableObservations:**

*Let $(x_0, y_0, z_0)$ be the solutions of (1). Then each of the following three triple of the non − zero distinct integers based on $x_0, y_0$ and $z_0$ aiso satisfies (1)*

*Triple 1:* $(x_0, y_n, z_n)$

$$x_n = 7^n x_0$$

$$y_n = \frac{1}{2i\sqrt{6}}[i\sqrt{6}(\alpha^n + \beta^n)y_0 - (\alpha^n - \beta^n)z_0]$$

$$z_n = \frac{1}{2i\sqrt{6}}[6(\alpha^n - \beta^n)y_0 + i\sqrt{6}(\alpha^n + \beta^n)z_0]$$

*where* $\alpha = 5 + i2\sqrt{6}, \quad \beta = 5 - i2\sqrt{6}$

*Triple 2:* $(x_n, y_0, z_n)$

$$x_n = \frac{1}{2\sqrt{10}}[\sqrt{10}(\alpha^n + \beta^n)x_0 + (\alpha^n - \beta^n)z_0]$$

$$y_n = y_0$$

$$z_n = \frac{1}{2\sqrt{10}}[10(\alpha^n - \beta^n)x_0 + \sqrt{10}(\alpha^n + \beta^n)z_0]$$

*where* $\alpha = 19 + 6\sqrt{10}, \quad \beta = 19 - 6\sqrt{10}$

*Triple 3:* $(x_n, y_n, z_0)$

$$x_n = \frac{1}{2}[(\alpha^n - \beta^n)x_0 - 3(\alpha^n - \beta^n)y_0]$$

$$y_n = \frac{1}{2}[5(\alpha^n - \beta^n)x_0 - (3\alpha^n - 5\beta^n)y_0]$$

$$z_n = z_0$$

$$where\ \alpha = 1,\ \beta = -1$$

## Conclusion

In this paper by introducing the linear transformations and employing the method of factorization, four different patterns of triples of non-zero distinct lattice points lying on the cone represented by the homogeneous ternary quadratic equation with three unknowns namely $z^2 = 10x^2 - 6y^2$ . are determined. Some remarkable relations between the integral solutions and special numbers are exhibited. Further, knowing a lattice point on the given cone , methods have been illustrated to generate a sequence of triples of lattice points satisfying the given cone.To conclude one may search for other methods of solutions and their corresponding properties.

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